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Modal Analysis of Structures with Holonomic Constraints

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A formulation is presented to predict the modal parameters of constrained structural systems. The formulation employs the Lagrange multiplier technique to investigate structural vibration before and after holonomic constraints are imposed. Holonomic constraint can either be of the linear, scleronomic type that accounts for structure compatibility, or of the rheonomic, nonlinear type for structures with deployable or articulated components. The vibration and stability of an integrated, constrained structural system can be predicted from the modal parameters of the corresponding unconstrained system and the characteristics of holonomic constraints.

Introduction

RECENT developments in aerospace technology have focused attention on the technical aspects of large-scale, complex, and constrained structural systems. Aircraft and spacecraft launch vehicles consist of a number of subsystems and components. Verification of structural design is typically performed using a limited series of tests on subsystem and component prototypes. To overcome the increasing size of the structural model and to ease the complexity of structural modification during the development phase, a method is needed wherein the vibration and stability of the integrated, constrained structural systems can be evaluated.

Large, complicated systems are often solved by either decomposition into components or subsystems and subsequent analysis, or system modeling with kinematic constraints. The former technique is similar to the so-called component mode synthesis method.[5] Previous research applied these techniques to mechanism dynamic analysis,[2] control system stability analysis,[3] and modal synthesis,[4] in which modal parameters, such as natural frequency and mode shape of each subsystem or component, are first determined. Modal analysis of an integrated, constrained structural system can then be conducted based on the predetermined modal parameters of the subsystems and components.

A number of subsystem and component coupling procedures have been proposed for solving undamped structural systems. Fuh and Chen[5] developed a procedure to construct a transformation matrix for formulating structure modal synthesis with constraints. Kuang and Tsuei[6] presented a model reduction procedure for substructure mode synthesis. Although both procedures require an orthogonal complement of the constraint Jacobiano matrix, the matrix construction is neither efficient nor unique. Moreover, most of the structure analyses to date consider only linear, scleronomic constraints that account for structure compatibility at the interface of two adjacent components. Little attempt has been made to include holonomic constraint at the interface or to predict its effect on the structure's natural frequencies when constraints are imposed.

Nonlinear and/or rheonomic constraints have been applied to the area of mechanism dynamics where numerical integration of system response is required.[7] Sohoni and Whitesell[8] developed a procedure for calculating the natural frequencies of a machine dynamics system. However, their formulation demands numerical identification of the state variables before the natural frequencies can be solved. Furthermore, the application is limited only to systems with scleronomic constraints. Modeling of structural systems with a deployable or articulated component, as well as structures with a coulomb friction boundary condition, require holonomic nonlinear constraints. Therefore, a method to predict the modal response of integrated, constrained structural systems is necessary.

The objective of this paper is to present a formulation for structural systems that incorporates holonomic, linear, and nonlinear constraints. The formulation employs the modal parameters of the unconstrained system and the Lagrange multiplier to account for the constraint force. Modal parameters of a constrained structural system can thus be predicted from those of the unconstrained system, as well as from constraint characteristics. Analysis and design of large, constrained structural systems require several iterations of system dynamics modification. This formulation allows one to predict the natural frequencies and stability of any linear, time-invariant system prior to numerical integration for response. Another important application of constrained systems modeling is the eigenstructure assignment,[9] since the constrained formulation provides a direct link between structural dynamics and control.

Constrained Equation Formulation

Consider a structural system modeled by a number of rigid and flexible bodies. An aircraft system, for example, can be modeled as a system composed of a rigid body (the fuselage) with flexible body subsystems (the lifting surfaces). The displacement of each subsystem or component can be analyzed in terms of deformable body mechanics. The equations of motion together with the associated boundary, initial, compatibility, and constitutive conditions, lead to a boundary value problem. Although a number of boundary value problems in continuous systems can be solved exactly, approximate methods are usually needed to obtain solutions for discrete systems. A continuous system can be approximated by the following equivalent discrete system:

$$M\ddot{q} + (D + G)\dot{q} + Kq + p(\dot{q}, q) = g(\dot{q}, q, t) \qquad (1)$$

where M, D, and K are $n \times n$ symmetric inertia, damping, and stiffness matrices, respectively; G is a skew-symmetric gyroscopic matrix; p is a high-order nonlinear term; q is a generalized coordinate vector; and g represents the external force.

Although the order of the constrained system increases with the introduction of the Lagrange multiplier, this technique is widely used because it provides information about the constraint force that is otherwise unavailable. When m-independent holonomic constraints, $\Phi(q, t) = 0$, are imposed on a structural system to account for either the boundary condition

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from assembly or a prescribed motion, the constrained system equations of motion become

$$M\ddot{q} + (D + G)\dot{q} + Kq + \Phi_q^T\mu + p(\dot{q}, q) = g(\dot{q}, q, t) \quad (2a)$$

$$\Phi_q\ddot{q} = -[(\Phi_q\dot{q})_q\dot{q} + 2\Phi_{qt}\dot{q} + \Phi_{tt}] \equiv r(\dot{q}, q, t) \quad (2b)$$

where the subscripts denote partial differentiation, and μ is the Lagrange multiplier vector. Note that the constraints are written as a set of algebraic equations, which are assumed to be twice differentiable in time. Equations (2a) and (2b) can be written in matrix form as

$$\begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \mu \end{bmatrix} = \begin{bmatrix} g + s \\ r \end{bmatrix} \quad (3)$$

where $s = -(D + G)\dot{q} - Kq - p(\dot{q}, q)$ represents the velocity and displacement dependent terms in Eq. (2a). Note that a unique solution to Eq. (3) exists if the constraints are linearly independent (i.e., Φ_q has full row rank). For initial conditions $q(0)$ and $\dot{q}(0)$, the solution $q(t)$, $\dot{q}(t)$, and $\mu(t)$ can be obtained by integrating Eq. (3). Numerical integration of Eq. (3) is straightforward if the constraints are continuous in $\dot{q}$, q, and t (Ref. 10). Note that μ is proportional to the constraint force.

For the state space representation with $x^T = [\dot{q}^T, q^T]$, the unconstrained system of Eq. (1) can be written as

$$\dot{x} = f(x, t) + Lg^* \quad (4)$$

Similarly, Eqs. (2a) and (2b) can be written as

$$\dot{x} = f(x, t) - J\Phi_x^T\mu + Lg^* \quad (5a)$$

$$\Phi_x I^*\dot{x} = r \quad (5b)$$

where

$$\Phi_x = [0\ \Phi_q], \qquad J = \begin{bmatrix} 0 & M^{-1} \\ 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$I^* = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \quad \text{and} \quad g^* = \begin{bmatrix} g \\ 0 \end{bmatrix}$$

If we denote $N = (\Phi_x I^* J\Phi_x^T)^{-1}$, the equation of motion with embedded constraint becomes

$$\dot{x} = (I - J\Phi_x^T N\Phi_x I^*)(f + Lg^*) + J\Phi_x^T Nr \quad (6)$$

A comparison of the unconstrained system in Eq. (4) to the corresponding constrained system in Eq. (6) reveals that not only is the state function in Eq. (6) altered by a coefficient matrix (the first term on the right-hand side), but also an additional excitation (the second term on the right-hand side) from the constraint is generated. The excitation in Eq. (6) arises from both g and r when the constraint is nonlinear and/or rheonomic. If $g = 0$, Eq. (5a) is then analogous to a dynamic system with control gain matrix $(-J\Phi_x^T)$ and input μ. Thus, a stability analysis of a constrained system is analogous to that of a control system. A similar idea has been developed for the indirect control of constrained dynamic systems.[11]

Dynamics of Constrained Systems

Previous analyses of the natural frequencies of both constrained structure systems[6] and machine dynamics systems[8] have been limited to computational procedures. Little attempt has been made to predict the changes in stability or the spectrum shift of a structure when constraints are imposed.

Consider $f(x, t)$ to be linear and time invariant; i.e., $f(x, t) = Ax$. Then define C_1 as

$$C_1 = M^{-1}\Phi_q^T(\Phi_q M^{-1}\Phi_q^T)^{-1}\Phi_q \quad (7)$$

and let

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Now the equations of motion with embedded constraint become

$$\dot{x} = (I - C)Ax + (I - C)Lg^* + \begin{bmatrix} M^{-1}\Phi_q^T(\Phi_q M^{-1}\Phi_q^T)^{-1}r \\ 0 \end{bmatrix} \quad (8)$$

Although the unconstrained system is linear and time invariant, the addition of a rheonomic and nonlinear constraint [$r \neq 0$ in Eqs. (2b) and (8)] generally results in a constrained system being nonlinear and time variant. Equation (8) can also be rewritten in second order form as

$$M\ddot{q} + (I - C_1^T)(G + D)\dot{q} + (I - C_1^T)Kq = \Phi_q^T(\Phi_q M^{-1}\Phi_q^T)^{-1}r + (I - C_1^T)g \quad (9)$$

Equation (9) represents a discrete constrained dynamic system under forced vibration in which the $\Phi_q^T(\Phi_q M^{-1}\Phi_q^T)^{-1}r$ term represents the generalized excitation arising from nonlinear and/or rheonomic constraints. Note that the stiffness matrix, $(I - C_1^T)K$, indicates that the spectrum of the constrained system has been shifted from that of the unconstrained system.

Let $\lambda_i(K)$ be the ith eigenvalue of K. The spectrum shift between the constrained and unconstrained systems can then be summarized as follows:

Lemma 1. For $\Phi_q \in R^{m\times n}$, rank$(\Phi_q) = m$, and $M = M^T$ with M positive definite $(M > 0)$, denote $C_1 = M^{-1}\Phi_q^T(\Phi_q M^{-1}\Phi_q^T)^{-1}\Phi_q$; then rank$(I - C_1) = n - m$ and $\lambda_i(I - C_1) = 0$ for $i = 1 \cdots m$ where $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

Lemma 2. For $K = K^T$, if $\lambda_i(K) > 0$, then $\lambda_i[(I - C_1)K] \ge 0$.

Lemma 3. For $K = K^T$, $\lambda_i(K) \le \lambda_{i+m}[(I - C_1)K] \le \lambda_{i+m}(K)$, for $i = 1 \cdots n - m$ and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

The proofs of Lemmas 1 and 2 are shown in the Appendix, and the proof of Lemma 3 is similar to that of the Rayleigh theorem of finite constraint.[12]

If the constraints are linear and scleronomic, then $r = 0$ in Eq. (2b). Thus, the homogeneous system (8) becomes

$$\dot{x} = (I - C)Ax \quad (10)$$

i.e.,

$$M\ddot{q} + (I - C_1^T)(G + D)\dot{q} + (I - C_1^T)Kq = 0 \quad (11)$$

Lemma 1 states that Eq. (11) has m zero eigenvalues associated with m constraints. The constrained system (10) or (11) is stable, provided that the unconstrained system (1) is stable. In addition, the natural frequencies of the constrained system are bounded below by those of the unconstrained system. Thus, application of linear, scleronomic constraints to a time-invariant structure system tends to make the system stiffer. A stable system remains stable when such constraints are imposed.

Consider a linear system with nonlinear, scleronomic constraints. For small vibration about an equilibrium, Eq. (2b) yields $r = 0$. One can then linearize Eq. (8) to obtain

$$\dot{x} = [I - C(0)]Ax \quad (12)$$

where $C(0)$ denotes C evaluated at the equilibrium. Thus, if the unconstrained system is stable, then with the addition of nonlinear, scleronomic constraints, the system will remain stable for small vibration about the equilibrium. In addition, when a system is subjected to both external excitation and rheonomic constraint [as in Eq. (8) or (9)], then, according to Lemma 3, the spectra of the constrained and unconstrained systems will be different. Consequently, the frequencies from $\boldsymbol{r}$, $\boldsymbol{g}$, and their combinations can lead to resonance. Note that scleronomic constraints can stabilize or destabilize the constrained system depending on the excitation frequency from $\boldsymbol{g}$. Similarly, the generalized force from the scleronomic or nonlinear constraint, $M^{-1}\Phi_q^T(\Phi_q M^{-1}\Phi_q^T)^{-1}r$ in Eq. (9), can either stabilize or destabilize the constrained system.

Equations (8) and (9) are direct formulations for constrained systems. The computational procedure is straightforward in that it requires no numerical identification. A singular value decomposition method can be applied to eliminate the zero frequencies associated with the constraints, as stated in Lemma 1. The stability and natural frequency variations can be predicted by the stability criteria developed by Yang and Mote.[13] Equations (8) and (9) can also be applied to modal synthesis and substructuring so that the constraints imposed between substructures can include general holonomic constraints to account for desired boundary conditions.

Examples

Stability of a Rotating Disk

The discrete model of a rigid disk with two linear spring-mass-damper oscillators rotating at Ω is used to ascertain the application of the constrained formulation. The rigid disk, pinned at the center, is capable of tilting transversely as shown in Fig. 1. A pair of spring-mass-oscillators rotating at Ω are positioned at a constant radius on the disk. The linearized equation of motion is written as[13]

$$\begin{bmatrix} \rho+I_t & 0 \\ 0 & \rho+I_t \end{bmatrix}\ddot{q} + \begin{bmatrix} c & 2\rho\Omega \\ -2\rho\Omega & c \end{bmatrix}\dot{q} + \begin{bmatrix} 1-\rho\Omega^2 & 0 \\ 0 & k-\rho\Omega^2 \end{bmatrix} q = 0 \tag{13}$$

where the generalized coordinate vector $q^T = [\theta, \psi]$; θ and ψ represent the tilting angles of the disk; I_t is the transverse moment inertia of the disk; ρ and c are the mass and damping coefficients, respectively, of each oscillator; and k is the stiffness ratio of the two moving oscillators ($k \geq 1$). For the case in which $c > 0$, two divergence instabilities are possible: $\Omega_{d_1} = (1/\rho)^{1/2}$ (so-called critical speed) and $\Omega_{d_2} = (k/\rho)^{1/2}$. Consider an additional constraint arising from the system's configuration requirement:

$$\Phi(q, t) \equiv \theta + a\psi + \sin \omega t = 0 \tag{14}$$

where a is real. Substituting Eq. (14) into Eq. (9), the constrained equation of motion becomes

$$\begin{bmatrix} \rho+I_t & 0 \\ 0 & \rho+I_t \end{bmatrix}\ddot{q} + \frac{1}{1+a^2}\begin{bmatrix} a^2 & -a \\ -a & 1 \end{bmatrix}\begin{bmatrix} c & -2\rho\Omega \\ 2\rho\Omega & c \end{bmatrix}\dot{q}$$

$$+\frac{1}{1+a^2}\begin{bmatrix} a^2 & -a \\ -a & 1 \end{bmatrix}\begin{bmatrix} 1-\rho\Omega^2 & 0 \\ 0 & k-\rho\Omega^2 \end{bmatrix} q$$

$$= \begin{bmatrix} 1 \\ a \end{bmatrix} \frac{\rho+1}{1+a^2}\,\omega^2 \sin \omega t \tag{15}$$

For the scleronomic constraint $\omega = 0$, the critical speed of the constrained system is

$$\Omega_{cr} = \left(\frac{a^2+k}{\rho(a^2+1)}\right)^{1/2}$$

which is bounded by the divergence of the unconstrained system (i.e., $\Omega_{d_1} \leq \Omega_{cr} \leq \Omega_{d_2}$). This result supports Lemma 3, indicating that a constrained system has a higher natural frequency than an unconstrained one. If the unconstrained system is rotating at subcritical speed (i.e., $\Omega < \Omega_{d_1}$) and is thus stable, the constrained system will also be stable for the same speed. As $a \to 0$, $\Omega_{cr} \to \Omega_{d_2}$; thus when $a = 0$, θ is constrained and the critical speed occurs at the divergence of ψ. Similarly, as $a \to \infty$, $\Omega_{cr} \to \Omega_{d_1}$, and divergence arises in θ. Note that the generalized excitation force from the constraint

$$\begin{bmatrix} 1 \\ a \end{bmatrix} \frac{\rho+1}{1+a^2}\,\omega^2 \sin \omega t$$

can excite resonance.

Consider the case in which $I_t = \rho = 1$ and $k = 2$. Figures 2 and 3 show the natural frequency of the constrained system vs rotation speed Ω for different constraint values of a in Eq. (14). Critical speed occurs when natural frequency drops to zero, indicating divergence. Both figures show critical speed decreasing from $\sqrt{2}$ to 1 when a increases from 0 to 100. Note that there is no stable supercritical speed region because of system damping. This example demonstrates the usefulness of Eq. (9) in predicting the stability and natural frequencies of constrained dynamic systems. Note that a rheonomic constraint

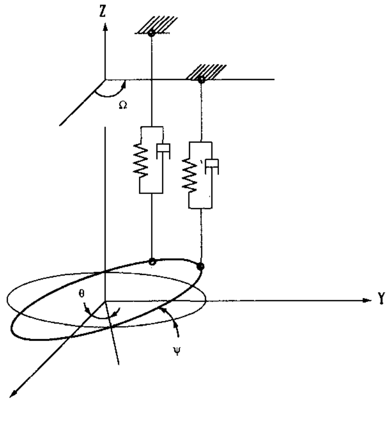

Fig. 1 Discrete model of a rigid disk with two moving loads.

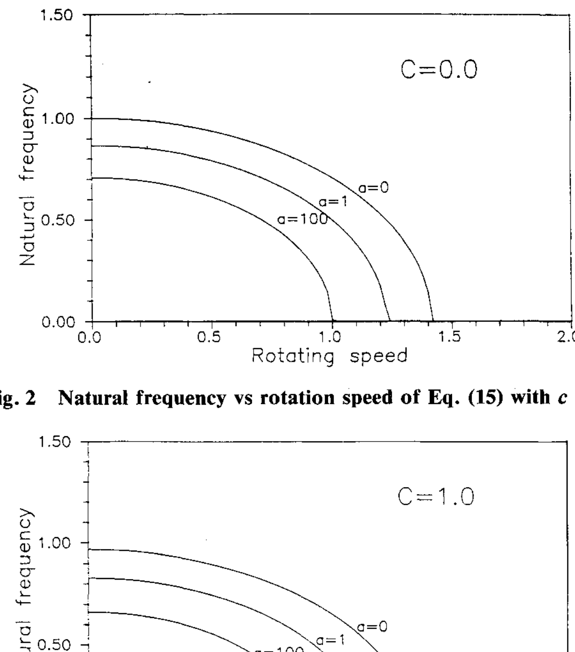

Fig. 2 Natural frequency vs rotation speed of Eq. (15) with $c = 0$.

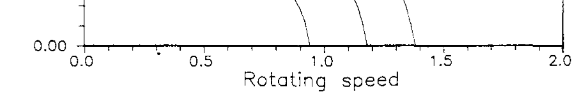

Fig. 3 Natural frequency vs rotation speed of Eq. (15) with $c = 1$.

can either stabilize or destabilize the dynamic system, depending on the excitation frequency of the external force and the constraint force.

Vibration of an Annular Plate Pinned at a Point

The example of an annular plate is employed in order to demonstrate the applicability of the constrained structure formulation. The annular plate is pinned at a point on its inner rim as shown in Fig. 4a. Consider only the out-of-plane vibration, the equation of motion of a classical thin plate of uniform thickness is

$$\nabla^4 u + \ddot{u} = 0 \tag{16}$$

The associated boundary conditions employed maintain that both bending moment and shear force are minimal at the inner and outer rims, r_a and r_b, except at the pinned point where displacement instead of shear force is zero:

$$B_1(u) \equiv \nabla^2 u - (1-\nu)\left(\frac{u_{,r}}{r} + \frac{u_{,\theta\theta}}{r^2}\right) = 0 \quad \text{at} \quad r = r_a, r_b \tag{17a}$$

$$B_2(u) \equiv (\nabla^2 u)_{,r} + (1-\nu)\frac{1}{r^2}\left(u_{,r} - \frac{u_{,\theta\theta}}{r}\right) = 0 \quad \text{at} \quad r = r_a, r_b \tag{17b}$$

where $u(r, \theta, t)$ denotes the transverse out-of-plane displacement in the cylindrical coordinates, and ν is the Poisson ratio. All terms used in Eqs. (16) and (17) are nondimensionalized.

Analysis of natural frequencies and mode shapes of the annular plate system is difficult because of the unique one-point-pinned boundary condition. An efficient, effective method of analysis is to employ the constrained structure formulation by using the natural frequencies and mode shapes of a free-free annular plate with the addition of a holonomic constraint. The transverse displacement of a free-free annular plate relative to its undeformed plane can thus be approximated by the modal expansion $u(r, \theta, t) = \boldsymbol{P}^T(r, \theta)\,\boldsymbol{\eta}(t)$, where $\boldsymbol{P}$ is a vector composed of the normal modes of a free-free annular plate, and

$$\boldsymbol{P}^T = [\sin(n\theta)R_{mn}(\omega_{mn}^{1/2} r) \cdots \cos(n\theta)R_{mn}(\omega_{mn}^{1/2} r) \cdots] \tag{18}$$

where

$$R_{mn}(\omega_{mn}^{1/2} r) = A_n J_n(\omega_{mn}^{1/2} r) + B_n K_n(\omega_{mn}^{1/2} r) + C_n I_n(\omega_{mn}^{1/2} r) + D_n L_n(\omega_{mn}^{1/2} r) \tag{19}$$

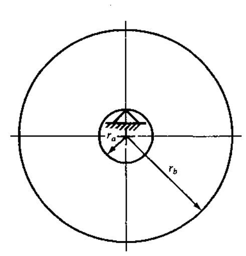

Fig. 4a Annular plate pinned at a point.

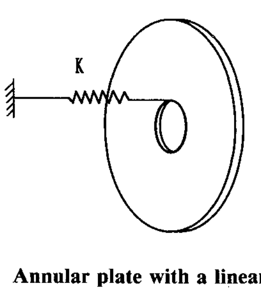

Fig. 4b Annular plate with a linear spring.

Table 1 Natural frequency of a free-free annular plate

Mode shape	Natural frequency
$(0, 2)^c$ and $(0, 2)^s$	5.03496
$(1, 0)$	8.35936
$(0, 3)^c$ and $(0, 3)^s$	12.35149
$(1, 1)^c$ and $(1, 1)^s$	19.01817

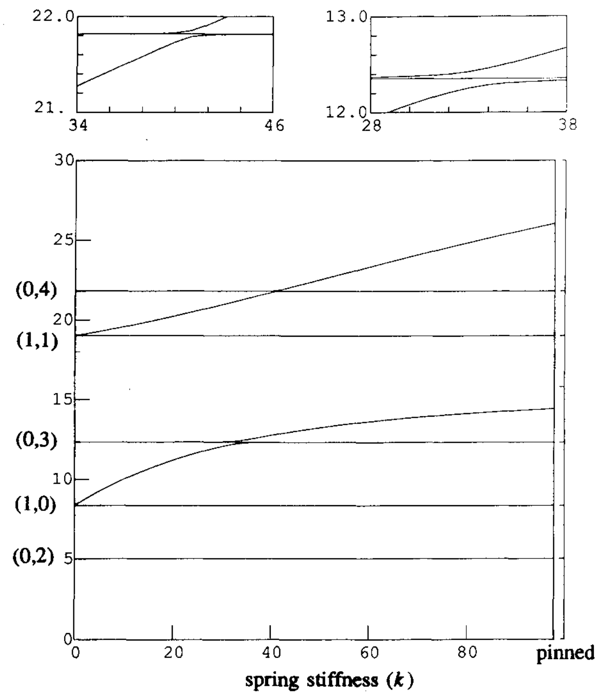

Fig. 5 Natural frequency loci of an annular plate for different spring stiffness.

The subscripts m and n in Eqs. (18) and (19) represent the numbers of nodal circle and nodal diameter, respectively. The functions J_n, K_n, I_n, and L_n are the nth-order Bessel functions and modified Bessel functions; A_n, B_n, C_n, and D_n are their corresponding coefficients determined by the solution of the eigenvalue problem. For each pair of (m, n), the two eigenfunctions, $\sin(n\theta)R_{mn}(\omega_{mn}^{1/2} r)$ and $\cos(n\theta)R_{mn}(\omega_{mn}^{1/2} r)$, correspond to the same frequency, ω_{mn}.

In a similar manner, the one-point-pinned constraint can be described by the modal expansion

$$\Phi \equiv u(r, \theta, t)|_{r = r_a, \theta = (\pi/2)} = 0 \tag{20}$$

Following from Eq. (9), the discretized equation of motion of a constrained annular plate is

$$\ddot{\boldsymbol{\eta}} + (\boldsymbol{I} - \boldsymbol{C}_1)\boldsymbol{K}\boldsymbol{\eta} = 0 \tag{21}$$

Note that with normal mode expansion the inertia matrix becomes identity and the stiffness matrix, $\boldsymbol{K} = \mathrm{diag}[\omega_{mn}^2]$, where ω_{mn} is the natural frequency of the free-free annular plate. The constraint coefficient matrix defined in Eq. (7) can now be written as

$$\boldsymbol{C}_1 = \frac{1}{\boldsymbol{P}^T\boldsymbol{P}}\boldsymbol{P}\boldsymbol{P}^T|_{r = r_a, \theta = (\pi/2)}$$

Consider an annular plate with $(r_a/r_b) = 1/4$. The natural frequencies of a free-free annular plate determined from Eqs. (16), (17a), and (17b) are listed in Table 1. The superscripts s and c in Table 1 represent $\sin(n\theta)$ and $\cos(n\theta)$ modes, respectively, and the (m, n) term refers to m nodal circles and n nodal diameters. For a modal expansion of seven modes, Eq. (19) is rearranged as

$$P^T = [(0, 2)^s\ (0, 3)^s\ (1, 1)^s\ (0, 2)^c\ (1, 0)\ (0, 3)^c\ (1, 1)^c]$$

By substituting the modal expansion into Eq. (21), the natural frequencies of an annular plate pinned at one point can be determined. To validate the constrained structure results, an annular plate system with a linear spring attached at the inner rim, as shown in Fig. 4b, is examined. In a limiting case, when the spring stiffness is large enough, this system should behave in a manner similar to that of an annular plate pinned at one point. Figure 5 shows the natural frequencies of the annular plate at different spring stiffness values. Note that the frequency of zero nodal circle modes $(0, n)$ is independent of the spring stiffness because the spring is located at the nodal diameters. However, nodal circle modes (1, 0) and (1, 1) bifurcate into two modes of different frequencies. The bifurcation branches that remain constant over all spring stiffness values represents the antisymmetric (cosine) mode, whereas the other, which increases with spring stiffness, represents the symmetric (sine) mode. The observation that all frequency loci asymptotically approach those of the constrained annular plate validates the aforementioned lemmas. Note that a structure's spectrum shifts upward when holonomic constraints are imposed and that, with or without constraint, the spectra are interlacing.

Conclusions

1) The stability and natural frequency variation of dynamic systems caused by imposed holonomic constraints can be predicted from Eqs. (8) or (9). For linear systems the natural frequencies corresponding to the constrained modes can be eliminated during eigenvalue calculation.

2) For a stable, initially unconstrained, free vibration system, if the added constraint is scleronomic, the constrained system remains stable. Application of linear, scleronomic constraints to a time invariant structural system tends to make the system stiffer. A stable system remains stable when constraints are imposed.

3) For systems with rheonomic constraint or with time-variant external excitation, the imposed holonomic constraint may either stabilize the system by shifting its spectrum away from the excitation and constraint frequencies or, conversely, destabilize the constrained system through resonance.

4) Application of holonomic constraints in the modeling of large and/or complex dynamic systems is evidenced in the area of dynamic analysis and modal synthesis. The present approach of formulating the constrained dynamic system is particularly useful in the modal analysis of structures with unusual or complicated boundary conditions. The example of an annular plate pinned at one point on its inner rim illustrates the constrained formulation application and demonstrates that present method's ability to predict the natural frequencies.

5) Another important application of the constrained system formulation is to the eigenstructure assignment. Equation (5a) is mathematically similar to the state equation formulation of feedback control systems. The natural frequency, vibration mode, and modal damping of a structure can be tuned to a desired range via imposed holonomic constraints or state feedback control. Application of the formulation can lead to a methodology for integrated design of structural control.

Appendix

Lemma 1. For $\Phi_q = R^{m\times n}$, rank$(\Phi_q) = m$, and $M = M^T$ where M is positive definite $(M > 0)$, if $C_1 = M^{-1}\,\Phi_q^T(\Phi_q M^{-1}\Phi_q^T)^{-1}\,\Phi_q$; then rank$(I - C_1) = n - m$ and $\lambda_i(I - C_1) = 0$ for $i = 1 \cdots m$ and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$.

Proof. Without loss of generality, let $M = I$; thus, $C_1 = \Phi_q^T(\Phi_q\Phi_q^T)^{-1}\,\Phi_q$. For Φ_q with full row rank, singular value decomposition shows that orthogonal matrices $U = R^{m\times m}$ and $V = R^{n\times n}$ exists such that $U^T\Phi_q V = \Sigma$ with

$$\Sigma = \begin{bmatrix} \Sigma_m & 0 \\ 0 & 0 \end{bmatrix}$$

and $\Sigma_m = \mathrm{diag}(\sigma_1, \ldots, \sigma_m)$ where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_m > 0$. Define the pseudoinverse

$$\Sigma^* = \begin{bmatrix} \Sigma_m^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} C_1 &= \Phi_q^T(\Phi_q\Phi_q^T)^{-1}\Phi_q \\ &= V\Sigma^T U^T\,(U\Sigma^{*T}\Sigma^* U^T)\,U\Sigma V^T \\ &= V\begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}V^T \end{aligned}$$

Thus,

$$I - C_1 = V\begin{bmatrix} 0 & 0 \\ 0 & I_{n-m} \end{bmatrix}V^T$$

and

$$\lambda_i(I - C_1) = 0,\ i = 1, \ldots, m$$

Lemma 2. For $K = K^T$, if $\lambda_i(K) > 0$, then $\lambda_i\,[(I - C_1)K] \ge 0$.

Proof. Since $K = K^T$, there exists an orthogonal matrix $U = R^{n\times n}$, such that $K = U\Lambda U^T$, where $\Lambda = \mathrm{diag}(\rho_1, \ldots, \rho_n)$ with $\rho_i \in R$. Without the loss of generality, let $I - C_1 = I - CC^T$. Thus, the eigenvalue problem $(I - CC^T)Ku = \lambda u$ becomes

$$(I - CC^T)U\Lambda U^T u = \lambda u \tag{A1}$$

Let $U^T u = v$ and $V = U^T C$, then Eq. (A1) yields

$$(I - VV^T)\Lambda v = \lambda v \tag{A2}$$

For $\lambda_i > 0$, $\Lambda = \Gamma\Gamma$, and $\Gamma v = w$, then Eq. (A2) becomes

$$(I - VV^T)\Gamma w = \lambda\Gamma^{-1}w \tag{A3}$$

Premultiply both sides of Eq. (A3) by $w^T\Gamma$:

$$w^T\Gamma^T(I - VV^T)\Gamma w = \lambda w^T w \tag{A4}$$

Since $(I - VV^T)$ is positive semidefinite, as shown in Lemma 1, $\lambda \ge 0$.

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